# THE BANACH-TARSKI PARADOX

Sidi Mohammed BOUGHALEM

sithlord-dev.github.io
University François Rabelais - Tours

The Banach-Tarski paradox is very well known in the mathematical world, it states that one can decompose a ball $A$ into pieces and use only rigid motions; mainly translations and rotations; to reconstruct; fr example, two copies of it. This seems impossible when one thinks of it since rigid motions are supposed to preserve volume, thus the name of "paradox" rather than theorem. This is mainly due to the way one 'decomposes' the said object. We shall prove the result in this paper in a pretty elementary way.

## 1 Some group theory

**Definition 1.1** (G-paradoxal, Equi-decomposability). *Let $G$ be a group, acting on a set $X$, $A, B$ subsets of $X$.*

1. *We say that $X$ is paradoxical under the action of $G$, if there exist pairwise disjoint subsets $A_1, \dots A_n$ and $B_1, \dots B_m$ of $A$ and $B$, respectively, and elements $g_1, \dots g_n, h_1, \dots h_m \in G$ such that*

$$X = \bigsqcup_{i=1}^{n} g_i(A_i) = \bigsqcup_{j=1}^{m} h_j(B_j)$$

2. *We say that $A$ and $B$ are equidecomposable, if there exist partitions $\{A_i\}$, $\{A_i\}$ of $A$ and $B$, respectively, and elements $g_1, \dots, g_n \in G$ such that*

$$A = \bigsqcup_{i=1}^{n} A_i, \quad B = \bigsqcup_{i=1}^{n} B_i, \quad B_i = g_i \cdot A_i \quad \forall i$$

   *One writes $A \sim B$.*
   *One notices that in this case, if $A$ is paradoxical, then so is $B$.*

3. *We say that a group is paradoxical if it is under its own action by left multiplication.*

**Example 1.2.** • *Every bounded interval of $\mathbb{R}$ is $\mathbb{R}$-paradoxical: Consider an interval $[a, b) \subset \mathbb{R}$, $c = \frac{a+b}{2}$ its center. One sees that*

$$[a, b[= [a, c[ \cup [c, b[ \quad \text{and} \quad [a, c[ \cap [c, b[= \emptyset$$

*Consider the isometry group $\mathbb{E}^1 = Isom(\mathbb{R}) = \{Ref_a : x \mapsto 2x - a, a \in \mathbb{R}\}$ that only consist of simple reflexions. $\mathbb{E}^1$ acts on $\mathbb{R}$, and one has:*

$$Ref_a([a, c[) = [a, b[= Ref_b([c, b[)$$

- $[a,b[\sim [a,b]$: *Let* $x_n \in [a,b]$ *such that* $x_1 = b$ *and let* $y_n \in \mathbb{R} \setminus [a,b]$ $\forall n \in \mathbb{N}$. *Consider*

$$\begin{aligned} g\colon \mathbb{R} &\to \mathbb{R} \\ z &\mapsto z && \forall z \neq x_n, y_n \\ x_n &\mapsto x_{n+1} && n \geq 1 \\ y_1 &\mapsto b \\ y_n &\mapsto y_{n-1} && n \geq 2 \end{aligned}$$

  $g$ *is bijective and one has* $g([a,b[) = [a,b]$.

The following result is in the heart of the commotion about this paradox, and may actually justify the seemingly intuitive 'absurdity' of such a result: The use of the axiom of choice.

**Lemma 1.3.** *Let* $G$ *be a group, acting freely on a set* $X$*. then if* $G$ *is paradoxical, so is* $X$*.*

*Proof.* Let $G$ be paradoxical, $\{(Ai, g_i), i \in \mathbb{N}\}$, $\{(B_j, h_j), j \in \mathbb{N}\}$ realising its decomposition. As $G$ acts on $X$, on has the orbit decomposition

$$X = \bigsqcup_{x \in X} \mathcal{O}_x$$

Using the axiom of choice, there exists a choice set $M$ that contains exactly one element from each orbit. Set $G \cdot M = \{g \cdot x / g \in G, x \in M\}$ and let

$$A'_i = \bigcup_{g_i \in A_i} \{g_i \cdot M\}, \quad B'_j = \bigcup_{h_j \in B_j} \{h_j \cdot M\}$$

We show that $\{(A'_i, g_i), i \in \mathbb{N}\}$, $\{(B'_j, h_j), j \in \mathbb{N}\}$ realise the decomposition of $X$. Suppose $\exists z \in A'_i \cap B'_j \neq \emptyset$

$$\exists x, y \in M \;\; z \in (A_i \cdot x) \cap (B_j \cdot y) \Rightarrow (A_i \cdot x) \cap (B_j \cdot y) \neq 0 \Rightarrow (G \cdot x) \cap (G \cdot y) \neq 0$$

$$\Rightarrow x = y \;\; \text{(By choice of M)}$$

Hence

$$\exists g_i \in A_i, \exists h_j \in B_j \;\; \text{such that} \;\; z = g_i \cdot x = h_j \cdot x \Rightarrow h_j^{-1} g_i \cdot x = x$$

$$\Rightarrow g_i = h_j \;\; \text{(As the action is free).}$$

Thus $A_i \cap B_j \neq \emptyset$ which is impossible. Moreover

$$\bigcup_{i=1}^{n} g_i \cdot A'_i = (\bigcup_{i=1}^{n} g_i \cdot A_i) \cdot M = G \cdot M = X = G \cdot M = (\bigcup_{i=1}^{n} h_j \cdot B_j) \cdot M = \bigcup_{j=1}^{n} h_j \cdot B'_j$$

which proves the claim. ∎

We will need the following lemma, known as the *Ping-pong* lemma:

**Lemma 1.4** (Ping-pong)**.** *[2] Let $G$ be a group acting on a set $X$, $H, K$ be two subgroups of $G$, with $|H| \geq 3$, $|K| \geq 2$. Suppose $G = \langle H \cup K \rangle$ and suppose there exists non-empty subsets $X_1, X_2$ of $X$ with $X_2 \nsubseteq X_1$ such that*

$$\forall h \in H \setminus \{e\} \quad h \cdot X_2 \subset X_1 \quad \textit{and} \quad \forall k \in K \setminus \{e\} \quad k \cdot X_1 \subset X_2$$

*Then*

$$G \cong H * K$$

*Proof.* By the universal property of the free product, one has a surjective group homomorphism (since $G$ is generated by $H \cup K$)

$$\begin{tikzcd} H \cup K \arrow[r, hook] \arrow[d] & G = \langle H \cup K \rangle \\ H * K \arrow[ur, dashed, two heads, "\phi"] & \end{tikzcd}$$

We show that it is an isomorphism, let $\omega$ be a reduced word from the alphabet of the (disjoint) union $H \setminus \{e\} \sqcup K \setminus \{e\}$. We need to show that its image under this isomorphism is not the identity.

- If $\omega = h_1 k_1 h_2 k_2 \dots k_{n-1} h_n$ then

$$\begin{aligned} \phi(\omega) \cdot X_2 &= h_1 k_1 \dots k_{n-1} h_n \cdot X_2 \\ &\subset h_1 k_1 \dots h_{n-1} k_{n-1} \cdot X_1 \\ &\subset h_1 k_1 \dots k_{n-2} h_{n-1} \cdot X_2 && (Ping) \\ &\subset h_1 k_1 \dots h_{n-3} k_{n-2} \cdot X_1 && (Pong) \\ &\dots \\ &\subset h_1 \cdot X_2 \subset X_1 \end{aligned}$$

  Since $X_2 \nsubseteq X_1$, $\phi(\omega) \neq e_G$.

- If $\omega = k_1 h_1 k_2 h_2 \dots h_{n-1} k_n$ then for $h \in H \setminus \{e\}$ : the previous result shows that $h\phi(\omega)h^{-1} \neq e_G$ and thus $\phi(\omega) \neq e_G$.

- If $\omega = h_1 k_1 h_2 k_2 \dots h_n k_n$ for $h \in H \setminus \{e, h^{-1}\}$ : the first result shows that $h\phi(\omega)h^{-1} \neq e_G$ and thus $\phi(\omega) \neq e_G$.

- If $\omega = k_1 h_1 k_2 h_2 \dots k_n h_n$ for $h \in H \setminus \{e, h^{-1}\}$ : the first result shows that $h\phi(\omega)h^{n} \neq e_G$ and thus $\phi(\omega) \neq e_G$.

$\square$

## 2 Main result

Let $\mathbb{E}^3 := Isom(\mathbb{R}^3)$ be the group of isometries acting on the euclidean space $\mathbb{R}^3$. Recall that a (the) free group of rank 2 consists of all words that can be built from the alphabet $\{a, b, a^{-1}, b^{-1}\}$. Note that

$$F_2 := \langle a, b \mid \rangle \cong \mathbb{Z} * \mathbb{Z}$$

We will show the main result of this paper, which translates in our actual set of definitions into:

**Theorem 2.1** (Banach-Tarski). *Every ball $\mathbb{B}_3 \subset \mathbb{R}^3$ is $\mathbb{E}^3$-paradoxical.*

To prove this result we will proceed by showing the following:

1. $F_2$ is paradoxical.
2. $F_2 \leq SO_3$ and $F_2$ acts freely on $\mathbb{S}^2 \setminus D$.
3. $\mathbb{S}^2 \setminus D \sim_{SO_3} \mathbb{S}^2$.
4. $\mathbb{B}_3 \setminus \{0\} \sim_{\mathbb{E}^3} \mathbb{B}_3$.

But before, and in order to understand how, geometrically, this paradoxicality happens, we will show a funny result that we would like to call, the *circle trick*:

**Theorem 2.2** (The circle trick). *Let $G = \mathbb{E}^2$, then*

$$\mathbb{S}^1 \setminus \{pt\} \sim_G \mathbb{S}^1$$

*Proof.* We identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Consider $\sigma$ to be a counter-clockwise rotation, say by angle $\frac{1}{5}$ radians around the origin. (i.e. the isometry $\sigma : z \mapsto e^{-\frac{1}{5}iz}$). As $2\pi$ is irrational, $\sigma(z)^n$ will never coincide with $z$. Now consider

$$A = \bigsqcup_{n \geq 1} \left\{\sigma^n(z) \,,\, z \in \mathbb{S}^1\right\} \quad \text{and} \quad \{pt\} = \left\{e^{i0} = 1\right\}$$

Then,

$$\sigma^{n-1}(z) \neq \sigma^{-1}(z) \;\; \forall n$$

Hence, by using the inverse rotation $\sigma^{-1} : z \mapsto e^{\frac{1}{5}iz}$ on the set $A$ and fixing the rest (i.e. $B = (\mathbb{S}^1 \setminus \{pt\}) \setminus A$) one recovers all the points since they are shifted $-\frac{1}{5}$ radian back:

$$\mathbb{S}^1 = B \sqcup A \sqcup \{pt\} = B \sqcup \sigma^{-1}(A)$$
$$\mathbb{S}^1 \setminus \{pt\} = B \sqcup A$$

Hence

$$\mathbb{S}^1 \setminus \{pt\} \sim_G \mathbb{S}^1$$

□

1. $F_2$ is paradoxical.

   Consider the following:

$$\begin{aligned} A &= \{a^n u, u \in F_2, n > 0\} \\ B &= \{b^n v, v \in F_2, n > 0\} \\ A' &= \{a^{-n} u, u \in F_2, n > 0\} \\ B' &= \{b^{-n} v, v \in F_2, n > 0\} \end{aligned}$$

   One clearly sees that

$$\begin{aligned} a^{-1}A &= \{au, bu, b^{-1}u, u \in B \cup B^{-1}\} \\ b^{-1}B &= \{bv, av, a^{-1}v, v \in A \cup A^{-1}\} \end{aligned}$$

   $F_2 = (A \sqcup B) \sqcup (A' \sqcup B' \sqcup \{e\})$ and

$$\begin{aligned} a^{-1}A \sqcup b^{-1}B &= F_2 \\ aA' \sqcup bB' \sqcup \{e\} &= F_2 \end{aligned}$$

2. $F_2 \leq SO_3$ and $F_2$ acts freely on $\mathbb{S}^2 \setminus D$.

   Let $h$ and $k$ be rotations of angle $arcos(3/5)$ that have orthogonal axes, i.e

$$h = \begin{pmatrix} \frac{3}{5} & \text{-}\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO_3 \quad \text{and} \quad k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & \text{-}\frac{4}{5} & \frac{3}{5} \end{pmatrix} \in SO_3.$$

   Let $H = \langle h \rangle$, $K = \langle k \rangle$ be subgroups of $SO_3$, and let $G = \langle H \cup K \rangle$. $G$ acts naturally on $\mathbb{R}^3$, while fixing

$$X = \left\{ \left( \frac{a}{5^k}, \frac{b}{5^k}, \frac{c}{5^k} \right)^{\mathsf{T}} \mid a, b, c \in \mathbb{Z}, k > 0 \right\}$$

   We restrict the action of $G$ on $X$ and consider the subspaces

$$\begin{aligned} X_1 &= \left\{ \left( \frac{3a \pm 4b}{5^k}, \frac{4a \pm 3b}{5^k}, \frac{c}{5^{k-1}} \right)^{\mathsf{T}} \mid a, b, c \in \mathbb{Z}, k > 0 \right\} \subset X \\ X_2 &= \left\{ \left( \frac{a}{5^{k-1}}, \frac{3b \pm 4c}{5^k}, \frac{4b \pm 3c}{5^k} \right)^{\mathsf{T}} \mid a, b, c \in \mathbb{Z}, k > 0 \right\} \subset X \end{aligned}$$

   Easy computations shows that

$$\begin{cases} \forall h \in H \setminus \{e\} & h \cdot X_2 \subset X_1 \\ \forall k \in K \setminus \{e\} & k \cdot X_1 \subset X_2 \end{cases} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in X_2 \setminus X_1$$

   Hence by Lemma 1.4

$$(F_2 \cong)\, H * K \cong G\, (\leq SO_3)$$

Now if we consider the action of $F_2$ over $\mathbb{S}^2$, every rotation has 2 fixed points, which are the intersections of its axis with $\mathbb{S}^2$, thus it is unfortunately not free and one can not directly use Lemma 1.3. However, one can make a slight change:
Consider $D := \{x \in \mathbb{S}^2 \mid \exists a \in F_2 : a \cdot x = x\}$. Then $\mathbb{S}^2 \setminus D$ is stable under the action of $F_2$, indeed for $\omega \in \mathbb{S}^2 \setminus D$ and $x \in D$ if $\omega(x) \in D$, then for $a \in F_2 \setminus \{e\}$

$$a \cdot \omega(x) = \omega(x) \Rightarrow \omega^{-1} \cdot a \cdot \omega(x) = x \Rightarrow \omega^{-1} \cdot a \cdot \omega = e \Rightarrow a = e$$

which is impossible, thus we have a free action of $F_2$ on $\mathbb{S}^2 \setminus D$.

3. $\mathbb{S}^2 \setminus D \sim_{SO_3} \mathbb{S}^2$.

This step is the core of the proof, basically we will find a way to reproduce the *circle trick* (Theorem 2.2) (where one exhibits a point out of $\mathbb{S}^1$, and uses the irrationality of the radius to rebuild the circle without that point) in a three-dimensional setting. We would like to find a rotation $\rho \in SO_3$ that would make the set $D$ 'disappear'. Since $D$ is countably infinite (and since there are uncountably many lines through the origin in $\mathbb{R}^3$) let $x \in \mathbb{S}^2 \setminus \{D \cup \text{-}D\}$ and let $d = (Ox)$, $r_\alpha$ be the rotation of angle $\alpha \in \left[0, \frac{\pi}{2}\right[$ and axis $d$. The key here is finding an angle $\theta$ such that no matter how many times we apply $\rho$ to any element of $D$ we can never land back in $D$. For $a \in D$, $n \in \mathbb{N}$ we define

$$A_{a,n} = \left\{\alpha \in \left[0, \frac{\pi}{2}\right[ \mid r_\alpha^n(a) \in D\right\}, \quad A = \bigcup_{a \in D} \bigcup_{n \geq 0} A_{a,n}$$

And thus, we want $\theta \in \left[0, \frac{\pi}{2}\right[ \setminus A$ which is possible assuming the axiom of countable choice (that states that the union of countably many countable sets is countable.) Hence one has that

$$r_\theta^n(D) \cap D = \emptyset \;\text{ and }\; r_\theta^n(D) \cap r_\theta^m(D) = \emptyset \;\; \forall m \neq n \in \mathbb{N}$$

We define

$$\overline{D} = \bigsqcup_{n \geq 0} r_\theta^n(D)$$

which is clearly disjoint since $D, r_\theta(D), r_\theta^2(D), \ldots$ are. One notices that

$$\overline{D} = D \bigsqcup_{n \geq 1} r_\theta^n(D) = \bigsqcup_{n \geq 0} r_\theta r_\theta^n(D) = D \sqcup r_\theta(\bigsqcup_{n \geq 0} r_\theta^n(D)) = D \sqcup r_\theta(\overline{D})$$

Finally, one has

$$\mathbb{S}^2 = (\mathbb{S}^2 \setminus \overline{D}) \sqcup D \sqcup r_\theta(\overline{D}) \Rightarrow \mathbb{S}^2 \setminus D = (\mathbb{S}^2 \setminus \overline{D}) \sqcup r_\theta(\overline{D})$$

Since $r_\theta(\overline{D}) \sim_{SO_3} \overline{D}$ we get

$$\mathbb{S}^2 \setminus D \sim_{SO_3} (\mathbb{S}^2 \setminus \overline{D}) \sqcup \overline{D} = \mathbb{S}^2$$

4. $\mathbb{B}_3 \setminus \{0\} \sim_{\mathbb{E}^3} \mathbb{B}_3$.

   Now choose a circle $S$ that has as missing point the origin of $\mathbb{B}_3$. By the circle trick (Theorem 2.2), we know that

   $$S \sim S \sqcup \{0\}_{\mathbb{B}_3}$$

   Hence

   $$\mathbb{B}_3 \setminus \{0\} = \mathbb{B}_3 \setminus (S \sqcup \{0\}) \sqcup S \sim \mathbb{B}_3 \setminus (S \sqcup \{0\}) \sqcup (S \sqcup \{0\}) = \mathbb{B}_3$$

*Proof of Theorem 2.1.* Now we put all the pieces together and prove our result. As $F_2$ is paradoxical (1), and every subgroup acts on the bigger group (say $G$) by left multiplication and without non-trivial fixed points (by inverses), $G$ is $F_2$-paradoxical, but then $G$ is also $G$-paradoxical. Thus every group that contains a free subgroup of index 2 is paradoxical. In particular, $SO_3$ is paradoxical.
By Lemma 1.3, one 'lifts' the paradoxicality to $\mathbb{S}^2 \setminus D$ as the action of $SO_3$ is free (2). Since $\mathbb{S}^2 \setminus D$ is equidecomposable to $\mathbb{S}^2$, $\mathbb{S}^2$ is $SO_3$-paradoxical (3): Here the 'lift' is used to reproduce the paradoxical decomposition of $F_2$ for $\mathbb{S}^2 \setminus D$ (in (1), one decomposes $F_2$ into two subgroups, each equidecomposable itself to $F_2$ and thus producing a 'doubling' effect)

$$\begin{aligned}\mathbb{S}^2 = \mathbb{S}^2 \setminus \{D\} \sqcup \{D\} &\sim \mathbb{S}^2 \setminus \{D\} \sqcup \mathbb{S}^2 \setminus \{D\} \sqcup \{D\} \\ &= \mathbb{S}^2 \setminus \{D\} \sqcup \mathbb{S}^2 \\ &\sim \mathbb{S}^2 \sqcup \mathbb{S}^2\end{aligned}$$

If we consider the radial correspondence

$$\mathbb{S}^2 \supset A \longmapsto A' = \bigcup_{a \in A} ]0, a] \subset \mathbb{B}_3 \setminus \{0\}$$

One clearly sees that $\forall n \in \mathbb{N}$, $A, B \subset \mathbb{S}^2$

$$(\bigsqcup_{n \geq 0} A_n)' = \bigsqcup_{n \geq 0} A'_n, \quad A' \cap B' = \emptyset \Leftrightarrow A \cap B = \emptyset$$

This way, the decomposition of $\mathbb{S}^2$ yields a decomposition of $\mathbb{B}_3 \setminus \{0\}$ which makes it paradoxical. Hence, by (4) we get that $\mathbb{B}_3$ is paradoxical and

$$\begin{aligned}\mathbb{B}_3 = \mathbb{B}_3 \setminus \{0\} \sqcup \{0\} &\sim \mathbb{B}_3 \setminus \{0\} \sqcup \mathbb{B}_3 \setminus \{0\} \sqcup \{0\} \\ &= \mathbb{B}_3 \setminus \{0\} \sqcup \mathbb{B}_3 \\ &\sim \mathbb{B}_3 \sqcup \mathbb{B}_3\end{aligned}$$

$\square$

# References


[1] Wagon, S. *The Banach-Tarski Paradox*, (Encyclopedia of Mathematics and its Applications), Cambridge University Press, 1985.

[2] de la Harpe, P. *Topics in Geometric Group Theory*, Bibliovault OAI Repository, the University of Chicago Press, 2000.